

Arithmetic of the [19,1,1,1,1] Fibration

by

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1. Introduction

Suppose $\pi : S \rightarrow \mathbb{P}^1$ is an elliptic $K3$ -surface over \mathbb{C} with a section, such that all its singular fibres are multiplicative. Suppose moreover that π is extremal, i.e., the group of sections of π is finite and the rank ρ of the Néron-Severi group $NS(S)$ of S is maximal, so $\rho = 20$. A formula of Shioda and Tate [Sh90, Cor. 5.3] then implies that π has precisely 6 singular fibers, consisting of n_i components ($i = 1, \dots, 6$) and $\sum n_i = 24$. A well-known result of Miranda and Persson [MP89] gives all 112 realizable 6-tuples $[n_1, n_2, n_3, n_4, n_5, n_6]$.

The largest number n_i appearing in this table is $n_i = 19$. In fact, T. Shioda [Sh03] showed for an elliptic $K3$ -surface over \mathbb{C} with a section that the property of containing a multiplicative fibre consisting of 19 components, determines it uniquely. Such a fibration was found explicitly by Y. Iron (a student of R. Livné) in his master's thesis [Ir03] and also by Shioda [Sh03], who realized the relation with work of M. Hall, jr. [H71]. The result can be stated as follows.

PROPOSITION 1.1 (Iron, Shioda). *Consider the polynomials*

$$f = t^8 + 6t^7 + 21t^6 + 50t^5 + 86t^4 + 114t^3 + 109t^2 + 74t + 28$$

and

$$g = t^{12} + 9t^{11} + 45t^{10} + 156t^9 + 408t^8 + 846t^7 + 1416t^6 + 1932t^5 \\ + 2136t^4 + 1873t^3 + (2517t^2 + 1167t + 299)/2.$$

Let S be the elliptic $K3$ -surface corresponding to the Weierstrass equation

$$y^2 = x^3 - 27f(t)x + 54g(t).$$

Then $S \rightarrow \mathbb{P}^1$ has precisely 6 singular fibres. The fibres over the zeroes of $4t^5 + 15t^4 + 38t^3 + 61t^2 + 62t + 59$ are of type I_1 . At $t = \infty$, the elliptic curve over $\mathbb{Q}(t)$ defined by

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the given Weierstrass equation has split multiplicative reduction. The fibre of $S \rightarrow \mathbb{P}^1$ over $t = \infty$ is of type I_{19} .

The fact that the reduction at $t = \infty$ is split multiplicative, is easily verified. Indeed, the change of variables $\xi := x/t^4$, $\eta := y/t^6$, $s := 1/t$ yields a Weierstrass equation where the special fibre at $s = 0$, is given as $\eta^2 = \xi^3 - 27\xi + 54 = (\xi - 3)^2(\xi - 3 + 9)$. This clearly defines a fibre with split multiplicative reduction. It follows that all 19 components of the fibre in S over $t = \infty$ are defined over \mathbb{Q} . Since by the formula of Shioda and Tate [Sh90, Cor. 5.3], these components together with the zero-section generate the Néron-Severi group of S , we obtain

COROLLARY 1.2. *The K3-surface S over \mathbb{Q} defined in Proposition 1.1 satisfies $\rho(S/\mathbb{Q}) = 20$, where $\rho(S/\mathbb{Q})$ denotes the rank of the subgroup $NS(S/\mathbb{Q})$ of the Néron-Severi group $NS(S)$ generated by the classes of \mathbb{Q} -rational divisors.*

It should be noted that this corollary contradicts [Sh94, Thm. 1]. Indeed, the argument presented in *loc.cit.* asserts that for almost all prime numbers p , the reduction $S \bmod p$ satisfies $\rho(S/\mathbb{F}_p) = 20$ or 22 . We will show that this is incorrect. More precisely, on the one hand, $\rho(X/\mathbb{F}_p) = 22$ is impossible for $p \neq 2$ and X/\mathbb{F}_p a K3-surface, due to an argument of M. Artin [A74, (6.8)]. On the other hand, we will see in the following (Cor. 2.2) that $\rho(S/\mathbb{F}_p) = 21$ for a set of primes with density $1/2$. As a consequence, the proof of Corollary 3 in [Sh94] is incomplete. This means that the following question is still open:

QUESTION 1.3. Is the maximal rank $rk E(\mathbb{Q}(t))$ for an elliptic curve $E/\mathbb{Q}(t)$ corresponding to an elliptic K3-surface over \mathbb{Q} equal to 18 or smaller than 18?

Recently, Shioda published an erratum [Sh05] to his paper [Sh94]. It turns out that several experts had pointed out the given mistake, including K. Hulek and H. Verrill. Interestingly, Shioda's original method of proof shows that $\rho(A/\mathbb{Q}) \neq 4 = h^{1,1}(A_{\mathbb{C}})$, when A/\mathbb{Q} is an abelian surface. In the rest of the present note, we will describe (Section 2) the representation of the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $H^2(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})$. This allows us to compute $\rho(S/\mathbb{F}_p)$ and the Hasse-Weil zeta function of S/\mathbb{F}_p , for almost all primes p , supplementing the remarks made by Shioda in [Sh05, § 2]. In Section 3, we recall a conjecture which Shioda formulated in [Sh03, § 4]. We use the surface S to illustrate how this conjecture can be verified in specific cases.

We remark as an aside, that similar examples may be found starting from other surfaces in the Miranda-Persson list [MP89]. Explicit equations for some of these surfaces can be obtained from [TY04] and [Sch04], and more recently for all of these surfaces from [BM04].

2. H^2 as a Galois module

Throughout, S is the elliptic surface over \mathbb{Q} defined in Proposition 1.1. Let $G_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. As a $G_{\mathbb{Q}}$ -module, the \mathbb{Q}_{ℓ} vector space $H^2(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_{\ell})$ splits as a direct sum into

the two-dimensional representation of the transcendental lattice and one-dimensional representations for the Néron-Severi group:

$$H^2(S_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell) = T_\ell \oplus \bigoplus_{j=1}^{20} \mathbb{Q}_\ell(1).$$

Here $\mathbb{Q}_\ell(1)$ denotes the one dimensional \mathbb{Q}_ℓ vector space on which a Frobenius conjugacy class at $p \neq \ell$ acts as multiplication by p . Our aim is to obtain the trace and the determinant of Frobenius conjugacy classes on T_ℓ .

For this purpose we have to restrict to the primes of good reduction, which in our case include all primes $p \neq 2, 3, 19$. We now discuss the three remaining primes.

For $p = 3$, a model for S over \mathbb{Q} with good reduction can be derived as follows. Consider the \mathbb{Q} -isomorphic surface coming from the equation

$$y^2 = x^3 - \frac{1}{3}f(t)x + \frac{2}{27}g(t)$$

and apply the change of variables $x \mapsto x + (t^4 + 3t^3 + 3t^2 + 4t + 4)/3$. This gives rise to the equation $y^2 = x^3 + A(t)x^2 + B(t)x + C(t)$ with

$$\begin{aligned} A &= t^4 + 3t^3 + 3t^2 + 4t + 4, \\ B &= 2t^6 + 8t^5 + 14t^4 + 22t^3 + 23t^2 + 14t + 4, \\ C &= t^8 + 5t^7 + 12t^6 + 21t^5 + 27t^4 + 24t^3 + 15t^2 + 5t + 1. \end{aligned}$$

The corresponding surface is easily seen to have good reduction at 3.

Concerning the reduction at 19, Shioda [Sh03, Thm. 3.1] shows that the minimal desingularization of the nodal Weierstrass surface over $\overline{\mathbb{F}}_{19}$ requires one further blow-up compared to characteristic 0. Reasoning differently, we will see below that the representation of $G_{\mathbb{Q}}$ on T_ℓ ramifies at 19. Hence the reduction at 19 is bad.

At $p = 2$, it turns out that no elliptic fibration on a $K3$ -surface in characteristic two containing an I_{19} -fibre exists. This follows from a direct calculation, by adopting techniques of W.E. Lang [La91, § 4] to the present situation. The details may be found in [Sch05].

We shall now compute the determinant of a Frobenius conjugacy class on T_ℓ for a good prime $p \neq 2, 19$. From [Sh03, § 3] (or by a direct calculation using the intersection matrix on $NS(S)$), one derives that the intersection form on the transcendental lattice of S is given by $\begin{pmatrix} 2 & 1 \\ 1 & 10 \end{pmatrix}$. This implies (compare [Li95, Example 1.6]) that the determinant of T_ℓ is given by

$$\det(\text{Frob}_p) = p^2 \varepsilon(p).$$

Here ε is the quadratic character corresponding to the extension $\mathbb{Q}(\sqrt{-19})/\mathbb{Q}$. Explicitly, $\varepsilon(p) = 1$ when $p \equiv 1, -2, -3, 4, 5, 6, 7, -8, \text{ or } 9 \pmod{19}$ and $\varepsilon(p) = -1$ for all other primes $p \neq 19$.

The trace of Frobenius elements acting on T_ℓ is related to the number of points in $S(\mathbb{F}_p)$ via the Lefschetz trace formula. Using the description of H^2 given above, this says

in our situation that

$$\#S(\mathbb{F}_{p^n}) = 1 + 20p^n + \text{trace}((\text{Frob}_p)^n | T_\ell) + p^{2n}.$$

Write $q = p^n$. Of the \mathbb{F}_q -rational points in S , precisely $19(q+1) - 19 = 19q$ are on the 19 components over $t = \infty$. The remaining ones are on the fibres over $t \in \mathbb{F}_q$, hence correspond to rational points on the projective cubic given by $y^2z = x^3 - 27f(t)xz^2 + 54g(t)z^3$. (For $p = 3$, we have to work with the other model given above.) This gives q rational points for $z = 0$. All other points are obtained from the affine Weierstrass equation $y^2 = x^3 - 27f(t)x + 54g(t)$. Their number is

$$\sum_{x,t \in \mathbb{F}_q} (1 + \chi_q(x^3 - 27f(t)x + 54g(t))) = q^2 + \sum_{x,t \in \mathbb{F}_q} \chi_q(x^3 - 27f(t)x + 54g(t))$$

where $\chi_q : \mathbb{F}_q \rightarrow \{0, \pm 1\}$ sends 0 to 0 and is the unique quadratic character on \mathbb{F}_q^* for $p \neq 2$. Combining the two formulas above yields

$$\text{trace}((\text{Frob}_p)^n | T_\ell) = -1 + \sum_{x,t \in \mathbb{F}_q} \chi_q(x^3 - 27f(t)x + 54g(t)).$$

Here is a small table computed using this formula. For convenience, we also include the trace of $(\text{Frob}_p)^2$, although we will not need it in the following.

p	trace Frob_p on T_ℓ	trace $(\text{Frob}_p)^2$ on T_ℓ
3	0	-18
5	-9	31
7	-5	-73
11	3	-233
13	0	-338
17	15	-353
23	-30	-158
29	0	-1682
31	0	-1922
37	0	-2738
41	0	-3362
43	-85	3527

According to [Li95, 1.3–1.5], the L -series determined by T_ℓ equals the L -series of a cusp form of weight 3 and nebentypus the quadratic character ε of conductor 19. We will now relate this cusp form to a Hecke character.

Let $K = \mathbb{Q}(\sqrt{-19})$ and denote by \mathbb{A}_K^* the ideles of K . Then

$$\psi_0(x) := x_\infty^{-2} \prod_{v \text{ finite}} \pi_v^{2v(x_v)}$$

(in which π_v generates the maximal ideal in $O_v \cap K$) defines the unique unramified Hecke character $\psi_0 : \mathbb{A}_K^*/K^* \rightarrow \mathbb{C}^*$ with ∞ -type 2. Using the table above, an argument completely analogous (in fact somewhat simpler since the field K has class number one) to [Li95, § 2] shows that the representations of $G_{\mathbb{Q}}$ given by $\text{Ind}_{G_K}^{G_{\mathbb{Q}}} \psi_0$ and by T_ℓ agree. Alternatively, this could be checked directly via the Faltings-Serre method [Se85]; in the present case, it suffices to check that the characteristic polynomials of Frobenius at the primes 3, 5, 7 and 31 agree.

Explicitly, this implies the following.

PROPOSITION 2.1. *Let $p \neq 2, 19$ be a prime number. The characteristic polynomial of Frob_p acting on T_ℓ is given by one of these rules:*

1. *If p is inert in $\mathbb{Q}(\sqrt{-19})$, then this polynomial equals $X^2 - p^2$.*
2. *If p splits in $\mathbb{Q}(\sqrt{-19})$, then write $4p = a^2 + 19b^2$ for integers a, b . Put $c := (a^2 - 19b^2)/2$. Then the required characteristic polynomial is $X^2 - cX + p^2$.*

The description of the characteristic polynomials here follows from the definition of the Hecke L -series $L(s, \psi_0)$. Its Euler factor at an inert prime of K is $(1 - \pi_v^2 p^{-2s})^{-1} = (1 - p^{2-2s})^{-1}$. The corresponding characteristic polynomial is $X^2 - p^2$. In case p splits in K , one can write $p = \pi \bar{\pi}$ for some $\pi = (a + b\sqrt{-19})/2$ in the ring of integers of K . The Euler factor now is $((1 - \pi^2 p^{-s})(1 - \bar{\pi}^2 p^{-s}))^{-1} = (1 - cp^{-s} + p^{2-2s})^{-1}$. This yields the other characteristic polynomials. \square

COROLLARY 2.2. *Let $p \neq 2, 19$ be a prime number. One has*

$$\rho(S/\mathbb{F}_p) = \begin{cases} 20 & \text{if } p \text{ splits in } \mathbb{Q}(\sqrt{-19}); \\ 21 & \text{if } p > 2 \text{ is inert in } \mathbb{Q}(\sqrt{-19}). \end{cases}$$

Indeed, for $p > 3$ this follows from the Tate conjecture. Since it is known for elliptic $K3$ -surfaces over finite fields of characteristic $p > 3$ [Ta94], the rank $\rho(S/\mathbb{F}_p)$ equals the multiplicity of the factor $X - p$ in the characteristic polynomial of Frob_p acting on $H^2(S_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)$. For the verification in case $p = 3$, we refer to the following example. \square

EXAMPLE 2.3. Consider the reduction S/\mathbb{F}_3 at 3 as constructed above. Our aim is to determine $NS(S/\overline{\mathbb{F}}_3)$. Since the singular fibre configuration in $S \rightarrow \mathbb{P}^1$ is the same in characteristic 3 as in characteristic 0, the Tate conjecture predicts $S/\overline{\mathbb{F}}_3$ to have Mordell-Weil group of rank 2 with the sections defined over \mathbb{F}_3 as a rank 1 subgroup. We find a section over \mathbb{F}_3 as

$$P = (t^2 + t, t - 1).$$

A further computation gives another section

$$Q = (-t^2 + (1 + i)t + 1, t^4 + it^3 + (1 + i)t - i)$$

over $\mathbb{F}_9 = \mathbb{F}_3[i]$ where $i^2 = -1$. In order to prove that these generate the Mordell-Weil group of the given fibration $S \rightarrow \mathbb{P}^1$ over $\overline{\mathbb{F}}_3$, we use the height pairing as defined by Shioda

in [Sh90]. An easy calculation (compare Section 3) gives

$$\langle P, P \rangle = \frac{6}{19}, \quad \langle P, Q \rangle = \frac{9}{19}, \quad \langle Q, Q \rangle = \frac{42}{19}.$$

As a consequence, the sublattice N of $NS(S/\overline{\mathbb{F}}_3)$ which is generated by P, Q and the trivial lattice consisting of the zero-section and the components of the fibres has discriminant

$$(-19) \det \left(\frac{3}{19} \begin{pmatrix} 2 & 3 \\ 3 & 14 \end{pmatrix} \right) = -9.$$

But by the general theory [A74, p. 556], a supersingular $K3$ -surface has discriminant $-p^{2\sigma_0}$ with $\sigma_0 \in \{1, \dots, 10\}$ the Artin invariant. Hence $N = NS(S/\overline{\mathbb{F}}_3)$ as claimed. In particular, the Tate conjecture holds for S in characteristic 3.

It is well known (see, e.g., [To89, Section 2.4]) that a 2-dimensional Galois representation as described here comes from a modular form. In the present case, this is the unique normalized newform of weight 3 for $\Gamma_0(19)$ with nebentypus the quadratic character ε of conductor 19, which has Fourier coefficients in \mathbb{Z} . Here, the uniqueness is shown as follows. Since the weight is odd, a newform with totally real Fourier coefficients has CM by its own nebentypus [R76, Proposition 3.3]. If the level of the modular form equals the discriminant of the CM field (up to sign), then the corresponding Hecke character has trivial conductor. If moreover the class number of the CM field is one, this Hecke character and hence the modular form is uniquely determined. Alternatively, in the present case the uniqueness may be read off from the tables [St].

Since all eigenvalues of Frobenius on H^2 are explicitly given, one obtains the local zeta function $Z(S/\mathbb{F}_p, T)$.

PROPOSITION 2.4. 1. *For primes $p > 2$ which are inert in $\mathbb{Q}(\sqrt{-19})$ one has*

$$Z(S/\mathbb{F}_p, T) = \frac{1}{(1-T)(1+pT)(1-pT)^{21}(1-p^2T)}.$$

2. *For primes p which split in $\mathbb{Q}(\sqrt{-19})$ one has*

$$Z(S/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-cT+p^2T^2)(1-pT)^{20}(1-p^2T)},$$

with c as defined in Proposition 2.1.2.

3. Reductions modulo supersingular primes

Modulo every odd prime p which is non-split in $\mathbb{Q}(\sqrt{-19})$, the surface S defines a supersingular elliptic $K3$ -surface over \mathbb{F}_p . For these primes, there is an injective reduction map $NS(S/\mathbb{Q}) \rightarrow NS(S/\overline{\mathbb{F}}_p)$ from a rank 20 lattice to a lattice of rank 22. The orthogonal complement of the image will be denoted by $L(p)$; it is a negative definite even lattice of rank two. In [Sh03, § 4], Shioda formulates the conjecture that such a lattice $L(p)$ should be similar to the transcendental lattice T_S of S/\mathbb{Q} . Recall that the intersection form on T_S

is given by $\begin{pmatrix} 2 & 1 \\ 1 & 10 \end{pmatrix}$. Here we will verify this conjecture for the reductions of S modulo 3 and modulo 19.

In characteristic 3, two extra generators of $NS(S/\overline{\mathbb{F}}_3)$ are given by the points P and Q defined in Example 2.3. Using a calculation as in the proof of [Sh90, Lemma 8.1] one finds that (in Shioda's notations)

$$\varphi(P) := (P) - (O) - 2F + \sum_{i=1}^4 \frac{14i}{19} \Theta_i + \sum_{i=5}^{18} \frac{5(19-i)}{19} \Theta_i$$

and

$$\varphi(Q) := (Q) - (O) - 2F + \frac{2}{19} \Theta_1 + \sum_{i=2}^{18} \frac{17(19-i)}{19} \Theta_i$$

generate the vector space $L(3) \otimes \mathbb{Q}$. Note that from this, the height pairings as given in Example 2.3 can be checked. In this vector space, $L(3)$ is generated by $19\varphi(Q)$ and $\varphi(P) - 7\varphi(Q)$. As a result, the intersection form on $L(3)$ can be given by $-3 \begin{pmatrix} 266 & -95 \\ -95 & 34 \end{pmatrix}$. This lattice is easily checked to be similar to T_S .

In characteristic 19, the equation $y^2 = x^3 - 27f(t)x + 54g(t)$ becomes

$$y^2 = x^3 - 8(t+3)(t-5)^7x - 3(t+3)^{11}(t-5).$$

It follows from [Sh03, Theorem 3.1] that the Mordell-Weil group is torsion-free of rank one, and is generated by

$$R := \left(2 \frac{(t+3)^{10}}{(t-5)^6}, 7\sqrt{-1} \frac{(t+3)^{15}}{(t-5)^9} \right).$$

Apart from the I_{19} -fibre over $t = \infty$, the elliptic fibration has one other reducible fibre: over $t = -3$ there is a fibre of type III . Its component not meeting the zero-section we denote by Θ . The lattice $L(19)$ is generated by Θ and $(R) - (O) - 5F$. On this basis, the intersection form is $-\begin{pmatrix} 2 & -1 \\ -1 & 10 \end{pmatrix}$. Hence also in this case, Shioda's conjecture holds.

Note that, in the notations of [Sh90], $\varphi(R) = (R) - (O) - 5F + \frac{1}{2}\Theta$. From this it can be seen, that the lattice generated by Θ and $2\varphi(R)$ is $\langle -2 \rangle \oplus \langle -38 \rangle$, which is *not* similar to the transcendental lattice.

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